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De Broglie-Type Relations from Nonlinear Evolution Equations

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De Broglie-type relations for the energy and momentum of the space-localized solutions to a class of nonlinear complex Hamiltonian evolution equations are derived without any "external assumptions." It is found that the Hermitian norm of the same solutions plays a more fundamental role than a mere normalization constant. The quantum commutator is obtained from the infinite-dimensional complex Poisson bracket, and it is in complete agreement with the above find. The significance that such relationships should exist, while being entirely independent of the concepts of pointlike particle and/or point charge, is briefly discussed.

1. INTRODUCTION

The adjective *space-localized* will appear many times in this paper. The meaning assigned to it for the present purposes is given by the following:

Definition. A singularity-free function $\psi = \psi(\mathbf{x}, t)$ of the coordinates x_i and the time t will be called space-localized (or localized) if $|\psi(\mathbf{x}, t)| \to 0$ sufficiently fast when $|\mathbf{x}| \to \infty$, so that its Hermitian norm $\langle\psi, \psi\rangle$ remains finite for all time:

$$\langle\psi, \psi\rangle = \int \psi^\dagger\psi \, d^3x = \sum_k \int \psi_k^*\psi_k \, d^3x < \infty \tag{1.1}$$

It is known that certain *nonlinear complex wave equations*, in one or more dimensions, possess space-localized solutions (which may be called *solitary waves*), including *solitons* (in the one-dimensional case). In a recent

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paper (Bodurov, 1996) it was shown that if *interaction terms* are introduced in those equations according to the rule

$$\frac{\partial}{\partial x^{\mu}} \to \frac{\partial}{\partial x^{\mu}} + igA_{\mu}, \qquad \mu = 0, 1, 2, 3 \tag{1.2}$$

where g is an arbitrary real constant, then (under certain conditions) the motion of the localized ψ-field, as a discrete entity, is identical to that of a point charge in an electromagnetic field whose 4-potential is A_{μ}. The same correspondence, although for a specific single equation, is obtained in Bialynicki-Birula and Mycielski (1976). Such results naturally lead to the following questions:

(a) If the motion of localized ψ-fields can represent the dynamics of point charges in electromagnetic fields, then can they represent also the de Broglie waves associated with point charges, or point masses?

(b) If the answer is yes, then are there de Broglie-type relations which connect the ψ-field energy (and momentum) to the frequency (and the wave-vector) of those waves?

(c) If the answer to the second question is yes, then what is the constant of proportionality in those de Broglie-type relations?

The objective of this paper is to present answers to the above questions (Section 3). Perhaps more importantly, it is to show that the unexpected answer to the third question is supported by at least one more independent argument (Section 4).

Despite the large number of works on nonlinear field theories, there seem to be only a few which address directly any of the above questions. The following brief review does not claim to be exhaustive:

Bialynicki-Birula and Mycielski (1976) investigated a nonlinear version of the Schrödinger (NLS) equation, in which the constant $\hbar$ appears explicitly and the nonlinear term is logarithmic. They discovered that such an equation admits closed-form space-localized solutions, which they called *gaussons*. Some of the results derived in the present paper (Section 3) for a class of nonlinear evolution equations may be found in that work derived specifically for the logarithmic NLS equation. For example, there it is shown that freely moving gaussons are de Broglie-type waves modulated by functions of the form $\exp[-(\mathbf{x} - \mathbf{v}t)^2/l^2]$.

Enz (1985) may have been the first to show that de Broglie-type relations can be derived from at least one nonlinear wave equation—that is, from a particular space-localized solution of the *sine-Gordon* equation in one space dimension, called the *breather.* The constant of proportionality in those relations (appearing in place of Planck's constant) is expressed solely in terms of the breather's parameters. The four conclusions to which Enz arrives are fully supported by the results of this paper. In addition, it is shown here that such relations are much more general, being valid for a class of *nonlinear Hamiltonian evolution* equations, and not only for the sine-Gordon equation.

Certainly, not all field equations which are Hamiltonian are expected to lead to de Broglie-type relations—rather, there are some from which such relations can be obtained. They belong to the family here called *complex Hamiltonian evolution* (CHE) equations. The next section will define them and present some of their properties, which will be used in Sections 3 and 4.

2. COMPLEX HAMILTONIAN EVOLUTION EQUATIONS

In this section it will be shown that:

(a) All systems of partial differential equations of the form [see equation (2.8)]

$$i\frac{\partial\psi_k}{\partial t} = \frac{\delta}{\delta\psi_k^*}\,\mathrm{H}[\boldsymbol{\psi}^*, \boldsymbol{\psi}], \qquad k = 1, 2, \ldots, r$$

are Hamiltonian evolution systems. Here $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_r)$ is an r-component field and $\boldsymbol{\psi}^*$ is its complex conjugate. $\delta/\delta\psi_k^*$ is the variational derivative of the real-valued functional

$$\mathrm{H}[\boldsymbol{\psi}^*, \boldsymbol{\psi}] = \int \mathcal{H}\left(\boldsymbol{\psi}^*, \boldsymbol{\psi}, \frac{\partial\boldsymbol{\psi}^*}{\partial x_i}, \frac{\partial\boldsymbol{\psi}}{\partial x_i}, \ldots\right) d^3x, \qquad i = 1, 2, 3$$

with respect to ψ_k^*. Such systems will be called *complex Hamiltonian evolution* (CHE) equations/systems.

(b) Associated with every CHE system there is an infinite-dimensional Poisson bracket [see equation (2.10)]

$$\{\mathrm{R}, \mathrm{S}\} = \frac{1}{i}\sum_{k=1}^{r}\int\left(\frac{\partial\mathrm{R}}{\partial\psi_k}\frac{\delta\mathrm{S}}{\delta\psi_k^*} - \frac{\delta\mathrm{S}}{\delta\psi_k}\frac{\delta\mathrm{R}}{\delta\psi_k^*}\right) d^3x$$

for the functionals $\mathrm{R} = \mathrm{R}[\boldsymbol{\psi}^*, \boldsymbol{\psi}]$ and $\mathrm{S} = \mathrm{S}[\boldsymbol{\psi}^*, \boldsymbol{\psi}]$.

(c) For all linear and nonlinear CHE systems/equations which are *gauge type I* invariant the Hermitian norm (1.1) is *constant* in time.

(d) Both the Schrödinger and Dirac equations are CHE equations.

To my knowledge, the above results have not been published previously. The reader not interested in how they were obtained and willing to accept them "as given" can omit this section.

The most general form of a system of Hamiltonian evolution equations for the components $u_k = u_k(\mathbf{x}, t)$ of the field $\mathbf{u}$ is

$$\frac{\partial u_k}{\partial t} = \sum_k \mathbf{J}_k\frac{\delta}{\delta u_k}\,\mathrm{H}[\mathbf{u}] \qquad (2.1)$$

where $\mathbf{J}_k$ are skew-adjoint differential operators, H[**u**] is the Hamiltonian functional for the field **u**, and $\delta/\delta u_k$ is the variational derivative with respect to u_k (Olver, 1993, Chapter 7).

However, the form (2.1) is too general for the purposes of this paper. We would like to have a form which is as specific as possible, and yet is general enough to contain as special cases the fundamental equations of quantum mechanics together with the nonlinear equations which in some respect are related to the quantum mechanical equations. Since the former are complex, such a form then has to represent a class of complex Hamiltonian evolution equations, both linear and nonlinear. It turns out that the desired form can easily be deduced by following the same process of extending a finite-dimensional real Hamiltonian system to an infinite-dimensional one [the process by which (2.1) was deduced (Olver, 1993, Chapter 7)], as shown next:

It is known that any Hamiltonian system

$$\frac{dq_n}{dt} = \frac{\partial}{\partial p_n} H(q, p), \qquad \frac{dp_n}{dt} = -\frac{\partial}{\partial q_n} H(q, p), \qquad n = 1, 2, \ldots, s \tag{2.2}$$

with a finite number of real canonical variables can be transformed into a more compact system of complex equations

$$\frac{dz_n}{dt} = -i \frac{\partial}{\partial z_n^*} H(z^*, z), \qquad \text{where} \quad \frac{\partial}{\partial z_n^*} = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial q_n} + i \frac{\partial}{\partial p_n} \right) \tag{2.3}$$

or

$$\frac{dz_n^*}{dt} = i \frac{\partial}{\partial z_n} H(z^*, z), \qquad \text{where} \quad \frac{\partial}{\partial z_n} = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial q_n} - i \frac{\partial}{\partial p_n} \right) \tag{2.4}$$

when its Hamiltonian is expressed as a function of the complex variables

$$z_n = \frac{1}{\sqrt{2}} (q_n + ip_n) \qquad \text{and} \qquad z_n^* = \frac{1}{\sqrt{2}} (q_n - ip_n) \tag{2.5}$$

The factor $1/\sqrt{2}$ which appears in these expressions is introduced only for convenience. In the above systems, z_n plays the role of the generalized momenta and its complex conjugate z_n^* that of the generalized coordinates. But such distinctions are not of much significance, for the roles can easily be exchanged. When $H(z^*, z)$ is a real-valued function of z_n and z_n^*, the two systems (2.3) and (2.4) are equivalent. Thus, one of them can be dropped without loss of information, showing that the complex Hamiltonian system consists of half as many equations as the real one.

Actually, a simple preliminary transformation must be applied to q_n and p_n so that the *dimensionalities* of the new variables q_n' and p_n' become the same. Otherwise, they cannot be combined into the complex variables z_n and z_n^*. The dimensionality of the products $q_n p_n$ is always the same—that of *action.* Hence, the transformation

$$q_n' = a_n q_n \qquad \text{and} \qquad p_n' = \frac{1}{a_n} p_n \qquad \text{with} \quad n = 1, 2, \ldots, s \tag{2.6}$$

will make the dimensionalities of q_n' and p_n' the same if the dimensionalities of the constants a_n are given by

$$\mathrm{dms}(a_n) = \sqrt{\mathrm{dms}(p_n)/\mathrm{dms}(q_n)}$$

Since this transformation does not alter the form of the Hamiltonian system (2.2), we can assume that it has already been applied when the complex transformation (2.5) is performed. This shows that the dimensionality of $z_k^* z_k$ is always that of *action.*

One finds easily that the transformation (2.5) "preserves" the Poisson bracket of any two real-valued or complex-valued functions R and S (the summation convention of repeated indexes applies to the rest of the paper):

$$\{R, S\} = \frac{\partial R}{\partial q_n} \frac{\partial S}{\partial p_n} - \frac{\partial S}{\partial q_n} \frac{\partial R}{\partial p_n} = \frac{1}{i} \left(\frac{\partial R}{\partial z_n} \frac{\partial S}{\partial z_n^*} - \frac{\partial S}{\partial z_n} \frac{\partial R}{\partial z_n^*} \right) \tag{2.7}$$

Thus, (2.5) can be considered to be a kind of *canonical transformation.*

To obtain a complex evolution equation describing the field $\psi = \psi(x_1, x_2, x_3, t)$, we start with the finite complex Hamiltonian system (2.3) and increase the number s of the generalized coordinates z_n to infinity. It is known how to perform such a transition when the coordinates are real-valued (Olver, 1993, Chapter 7). We extend the same procedure to the complex case by replacing (a) the set of all coordinates z_n with the *ψ-field,* and the z_n^* with the ψ^*-field, (b) the ordinary time derivative d/dt with the partial time derivative $\partial/\partial t$, (c) the Hamiltonian function $H(z^*, z)$ with the *Hamiltonian functional* $\mathrm{H}[\psi^*, \psi]$, and (d) the partial derivatives $\partial/\partial z_n$, $\partial/\partial z_n^*$ with the *variational derivatives* $\delta/\delta\psi_k$, $\delta/\delta\psi_k^*$, where $\psi = (\psi_1, \ldots, \psi_r)$ is an r-component field and ψ^* is its complex conjugate. The result is a system of *r complex Hamiltonian evolution* equations, one for each component of the ψ-field:

$$\frac{\partial \psi_k}{\partial t} = -i \frac{\delta}{\delta \psi_k^*} \mathrm{H}[\psi^*, \psi] = -iH_k(\psi^*, \psi), \qquad k = 1, 2, \ldots, r \tag{2.8}$$

The Hamiltonian functional H is an integral of some real-valued *Hamiltonian density* $\mathcal{H}$ over all space (all functionals will be denoted with roman capital letters, all densities with script capital letters, and all operators with bold italic capital letters):

$$\mathrm{H}[\psi^*, \psi] = \int \mathcal{H}(\psi^*, \psi, \partial_i\psi^*, \partial_i\psi, \ldots)\, d^3x, \qquad i, j = 1, 2, 3$$

where ∂_i stands for the derivative with respect to the space coordinate x_i. $\mathcal{H}$ may contain derivatives of order higher than the first. Accordingly, the variational derivative of H with respect to ψ_k^* is given by the *Euler–Lagrange expressions*

$$\frac{\delta \mathrm{H}}{\delta \psi_k^*} = \frac{\partial \mathcal{H}}{\partial \psi_k^*} - \sum_i \frac{d}{dx_i} \frac{\partial \mathcal{H}}{\partial(\partial_i \psi_k^*)} + \sum_{i,j} \frac{d^2}{dx_i\, dx_j} \frac{\partial \mathcal{H}}{\partial(\partial_{ij} \psi_k^*)} - \cdots = \boldsymbol{H}_k(\psi^*, \psi) \tag{2.9}$$

where $\boldsymbol{H}_k(\psi^*, \psi)$ are operators acting on ψ, nonlinearly in general.

The expression for the *infinite-dimensional complex Poisson bracket* of two functionals $\mathrm{R} = \mathrm{R}[\psi^*, \psi]$ and $\mathrm{S} = \mathrm{S}[\psi^*, \psi]$ is defined by performing the transition $s \to \infty$ (a finite to an infinite set of generalized coordinates) on the finite-dimensional complex Poisson bracket (2.7) (both finite- and infinite-dimensional Poisson brackets will be denoted by the same symbol $\{\cdot, \cdot\}$, since it will be clear from the context to which type it refers):

$$\{\mathrm{R}, \mathrm{S}\} = \frac{1}{i} \int \left(\frac{\delta \mathrm{R}}{\delta \psi_k} \frac{\delta \mathrm{S}}{\delta \psi_k^*} - \frac{\delta \mathrm{S}}{\delta \psi_k} \frac{\delta \mathrm{R}}{\delta \psi_k^*} \right) d^3x \tag{2.10}$$

H is routine to verify that (2.10) satisfies all the requirements for a Poisson bracket. This is an exact form-replica of the *classical* infinite-dimensional Poisson bracket

$$\{\mathrm{R}, \mathrm{S}\}_{\mathrm{CL}} = \int \left(\frac{\delta \mathrm{R}}{\delta \eta_k} \frac{\delta \mathrm{S}}{\delta \pi_k} - \frac{\delta \mathrm{S}}{\delta \eta_k} \frac{\delta \mathrm{R}}{\delta \pi_k} \right) d^3x \tag{2.11}$$

where $\eta_k = \eta_k(\mathbf{x}, t)$ and $\pi_k = \pi_k(\mathbf{x}, t)$ are the components of two real, canonically conjugate fields. However, the bracket (2.10) has an important advantage over the latter. It is not evident how the latter could be made Lorentz-invariant, since the definition of the fields η_k and π_k depends on the frame of reference. Clearly, this criticism does not apply to the complex Poisson bracket (2.10). Details and a discussion of the difficulties with (2.11) can be found in Goldstein (1980; Section 12.14).

The time derivative of any functional $\mathrm{R} = \mathrm{R}[\psi^*, \psi, t]$ is given in terms of the complex Poisson bracket, as in the real finite-dimensional case, by

$$
\begin{aligned}
\frac{d\mathrm{R}}{dt} &= \int \left(\frac{\delta \mathrm{R}}{\delta \psi_k} \frac{\partial \psi_k}{\partial t} + \frac{\delta \mathrm{R}}{\delta \psi_k^*} \frac{\partial \psi_k^*}{\partial t} \right) d^3x + \frac{\partial \mathrm{R}}{\partial t} \\
&= \frac{1}{i} \int \left(\frac{\delta \mathrm{R}}{\delta \psi_k} \frac{\delta \mathrm{H}}{\delta \psi_k^*} - \frac{\delta \mathrm{H}}{\delta \psi_k} \frac{\delta \mathrm{R}}{\delta \psi_k^*} \right) d^3x + \frac{\partial \mathrm{R}}{\partial t} = \{\mathrm{R}, \mathrm{H}\} + \frac{\partial \mathrm{R}}{\partial t}
\end{aligned}
\tag{2.12}
$$

If the Hamiltonian functional $\mathrm{H}[\boldsymbol{\psi}^*, \boldsymbol{\psi}]$ does not depend explicitly on the time, then it is constant, as seen from (2.12).

It is simple to verify that equations (2.8) can also be obtained as the conditions, i.e., the Euler–Lagrange equations, for the extremum of the functional

$$
\int \mathrm{L}[\boldsymbol{\psi}^*, \boldsymbol{\psi}]\, dt = \iint \mathcal{L}'\, d^3x\, dt = \iint \left(i\boldsymbol{\psi}^\dagger \frac{\partial \boldsymbol{\psi}}{\partial t} - \mathcal{H} \right) d^3x\, dt
$$

where $\boldsymbol{\psi}^\dagger = (\boldsymbol{\psi}^*)^t$ is the transpose of the complex conjugate of $\boldsymbol{\psi}$. This establishes that $\mathrm{L}[\boldsymbol{\psi}^*, \boldsymbol{\psi}]$ is the *Lagrangian functional*, and that the relation of the *Lagrangian density* $\mathcal{L}'$ of the $\boldsymbol{\psi}$-field to its *Hamiltonian density* $\mathcal{H}$ is $\mathcal{L}' = i\boldsymbol{\psi}^\dagger\, \partial\boldsymbol{\psi}/\partial t - \mathcal{H}$. The same equations (2.8) are obtained if we start with the complex conjugate $\mathcal{L}'^*$. Hence, we can define a real Lagrangian density by $\mathcal{L} = (\mathcal{L}' + \mathcal{L}'^*)/2$,

$$
\mathcal{L} = \frac{i}{2} \left(\boldsymbol{\psi}^\dagger \frac{\partial \boldsymbol{\psi}}{\partial t} - \frac{\partial \boldsymbol{\psi}^\dagger}{\partial t} \boldsymbol{\psi} \right) - \mathcal{H} \tag{2.13}
$$

which makes the Lagrangian functional also real-valued. The last relation allows us to give simple proofs to the following two propositions:

Proposition 1. The $\boldsymbol{\psi}$-field energy density, i.e., the $\mathcal{T}_{00}$ component of the stress-energy four-tensor, is equal to the Hamiltonian density $\mathcal{H}$.

Proof.

$$
\mathcal{T}_{00} = \frac{\partial \eta_i}{\partial t} \frac{\partial \mathcal{L}}{\partial(\partial_t \eta_i)} - \mathcal{L} = \frac{i}{2} \left(\psi_k^* \frac{\partial \psi_k}{\partial t} - \frac{\partial \psi_k^*}{\partial t} \psi_k \right) - \mathcal{L} = \mathcal{H} \tag{2.14}
$$

where the field $\boldsymbol{\eta}$ has $2r$ components, with $i = 1, 2, \ldots, 2r$,

$$
\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_{2r}) = (\psi_1, \ldots, \psi_r, \psi_1^*, \ldots, \psi_r^*) \quad \blacksquare
$$

Most of what follows depends on the validity of identifying the $\boldsymbol{\psi}$-field total energy with the value of the Hamiltonian functional for that field (within an additive constant). This proposition assures us that such an identification is well justified, and that it is not a mere assumption.

Proposition 2. The Hermitian norm

$$\langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle = \int \boldsymbol{\psi}^{\dagger} \boldsymbol{\psi} \, d^3x \tag{2.15}$$

is a first integral of all CHE equations whose Hamiltonian density $\mathcal{H}$ is invariant under the gauge transformation of the first type

$$\psi_k' = \psi_k \, e^{i\epsilon} \qquad \text{and} \qquad \psi_k'^{*} = \psi_k^{*} e^{-i\epsilon} \tag{2.16}$$

where ϵ is the transformation parameter and $k = 1, \ldots, r$.

Proof. From (2.13) we see that if $\mathcal{H}$ is gauge type I invariant, then so is the Lagrangian density $\mathcal{L}$. Hence, according to Noether's theorem, the following conservation law holds:

$$\frac{\partial \mathcal{D}}{\partial t} + \nabla \cdot \mathcal{F} = 0 \tag{2.17}$$

where the *density* $\mathcal{D}$ and the *flux* $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ are given by

$$\mathcal{D} = i \left(\psi_k^{*} \frac{\partial \mathcal{L}}{\partial(\partial_t \psi_k^{*})} - \frac{\partial \mathcal{L}}{\partial(\partial_t \psi_k)} \psi_k \right), \qquad \mathcal{F}_i = i \left(\psi_k^{*} \frac{\partial \mathcal{L}}{\partial(\partial_i \psi_k^{*})} - \frac{\partial \mathcal{L}}{\partial(\partial_i \psi_k)} \psi_k \right) \tag{2.18}$$

Here, for economy of writing, it was assumed that $\mathcal{H}$ contains only first-order space derivatives which does not affect the generality of the proof. Using the expression (2.13) in (2.18), one obtains

$$\mathcal{D} = \psi_k^{*} \psi_k = \boldsymbol{\psi}^{\dagger} \boldsymbol{\psi} \tag{2.19}$$

$$\frac{\partial}{\partial t} \langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle = \frac{\partial}{\partial t} \int \boldsymbol{\psi}^{\dagger} \boldsymbol{\psi} \, d^3x = -\int \nabla \cdot \mathcal{F} \, d^3x = 0 \quad \blacksquare \tag{2.20}$$

Alternatively, one can use the relation (2.12) with $\mathrm{R} = \langle \boldsymbol{\psi}, \boldsymbol{\psi} \rangle$, then take into consideration the invariance of $\mathcal{H}$ under (2.16), and arrive at the same conclusion, without relying on Noether's theorem.

Both the Schrödinger and Dirac equations are of the form

$$i \frac{\partial \psi_k}{\partial t} = \boldsymbol{H}_{kj} \, \psi_j \qquad \text{or} \qquad i \frac{\partial \boldsymbol{\psi}}{\partial t} = \boldsymbol{H} \boldsymbol{\psi}, \qquad k, j = 1, \ldots, r \tag{2.21}$$

This form is derivable via the rule (2.8) from real-valued bilinear Hamiltonian functionals

$$\mathrm{H}[\boldsymbol{\psi}^{*}, \boldsymbol{\psi}] = \int \psi_k^{*} \boldsymbol{H}_{kj} \, \psi_j \, d^3x = \int \boldsymbol{\psi}^{\dagger} \boldsymbol{H} \boldsymbol{\psi} \, d^3x \tag{2.22}$$

where the linear operators H_{kj} are the elements of the matrix differential operator H, which is Hermitian because $\mathrm{H}[\psi^*, \psi]$ is real-valued. Hence, the Schrödinger and Dirac equations belong to the family of linear CHE equations, which shows that (2.8) is of the desired type.

It should be noticed that in general, a solution of a CHE equation cannot be normalized and remain a solution of the same equation (as is the case in quantum mechanics), since the CHE equations are not necessarily linear. Consequently, the quantity $\langle \psi, \psi \rangle$ is not dimensionless, and in fact its dimensionality is always that of *action.* This follows from Proposition 1 and the general form (2.8) of the CHE equations, or by observing that the dimensionality of $z_n^* z_n$ in a finite complex Hamiltonian system is always that of *action,* as noted earlier.

3. DE BROGLIE-TYPE RELATIONS

The nature of the de Broglie relations tells us that the only evolution equations from which one may expect relations of this type [as defined in question (b) of Section 1] are those which are derivable from a Hamiltonian density, so that the conservation of energy is guaranteed for an isolated system by Proposition 1, and which, in addition, are either *Galilei-invariant* or *Lorentz-invariant.* Only the Galilei invariance case and solutions with a single region of localization will be considered in this paper. The case of Lorentz invariance will be left for a subsequent paper.

It is known that the nonlinear CHE equations of the form

$$i \frac{\partial \psi}{\partial t} = \mu \nabla^2 \psi + G'(\psi^\dagger \psi)\psi \tag{3.1}$$

where μ is an unspecified real constant and ψ is complex in general, are Galilei-invariant and that some of them possess space-localized solutions, including solitons (in the case of one space dimension). All such equations are derivable from the Hamiltonian functional

$$\mathrm{H} = \int \mathcal{H}\, d^3x = \int \left(-\mu \nabla \psi_k^* \cdot \nabla \psi_k + G(\psi^\dagger \psi) \right) d^3x \tag{3.2}$$

where $G'(\rho) = dG\rho)/d\rho$ and $G(\rho)$ is any function such that the stationary solutions $\psi(\mathbf{x}, t) = \varphi(\mathbf{x}) \exp(-i\omega_0 t)$ of (3.1) have a finite norm

$$\langle \psi, \psi \rangle = \int \psi^\dagger \psi \, d^3x = \int \varphi^2 \, d^3x < \infty$$

which in turn assures the localization of the ψ-field. Here, the field φ is real, but not necessarily scalar, and $\varphi^2 = \Sigma_{k=1}^r \varphi_k^2$.

Let the stationary ψ-field be observed from a coordinate system moving with a velocity $-\mathbf{v}$ relative to the original system, so that in the new system the velocity of the field's region of localization is $+\mathbf{v}$. Then the solution must be transformed as

$$\psi'(\mathbf{x}, t) = \varphi(\mathbf{x} - \mathbf{v}t)\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} = \varphi' e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{3.3}$$

where ω is the new frequency and $\mathbf{k} = (k_1, k_2, k_3)$ is an as-yet-unspecified vector, both depending on the transformation parameters $\mathbf{v} = (v_1, v_2, v_3)$. By substituting this ansatz into equation (3.1) and observing that both $\varphi = \varphi(\mathbf{x})$ and $\varphi' = \varphi(\mathbf{x} - \mathbf{v}t)$ are solutions of

$$\omega_0\, \varphi = \mu\nabla^2\varphi + \varphi\, G'(\varphi^2)$$

one finds that the conditions

$$\mathbf{v} = -2\mu\mathbf{k} \qquad \text{and} \qquad \omega - \omega_0 = -\mu k^2 \tag{3.4}$$

must hold if (3.3) is to be a solution of equation (3.1). Then, if we have to associate a wave with the space-localized ψ-field, (3.3) tells us that when the conditions (3.4) are met, $\exp i(\mathbf{k}\cdot\mathbf{x} - \omega t)$ is such a wave, modulated by the function $\varphi(\mathbf{x} - \mathbf{v}t)$. This simple identification has remarkable implications, which will be briefly discussed in Section 5.

Now it will be shown, without relying on an "external assumptions," that ω and $\mathbf{k}$ satisfy de Broglie-type relations in which the energy and the momentum are those of the ψ-field, instead of those of some fictitious pointlike particle. When $\mathbf{k}$ is eliminated from the expressions (3.4), we see that the change of frequency is quadratic in $\mathbf{v}$,

$$\omega - \omega_0 = -\frac{v^2}{4\mu} \tag{3.5}$$

just as the *kinetic energy* of a material object is quadratic in its velocity. By Proposition 1 the ψ-field energy is given by the value of the Hamiltonian functional (3.2) (within an additive constant, which is irrelevant for this derivation)

$$\mathrm{E} = \int \mathcal{H}[\psi^*, \psi]\, d^3x$$

when the specific solution ψ is inserted in it. Then equation (3.1) can be used to eliminate all space derivatives in E as follows:

$$\begin{aligned}\mathrm{E} &= \int (-\mu\nabla\psi_k^*\cdot\nabla\psi_k + G)\, d^3x \\ &= \frac{\mu}{2}\int (\psi^\dagger\nabla^2\psi + \nabla^2\psi^\dagger\, \psi)\, d^3x + \int G\, d^3x\end{aligned}$$

Hence

$$\mathrm{E} = \frac{i}{2}\int\left(\psi^{\dagger}\frac{\partial\psi}{\partial t} - \frac{\partial\psi^{\dagger}}{\partial t}\psi\right)d^3x + \int(G - |\psi|^2G')\,d^3x \tag{3.6}$$

The intermediate step was necessary to ensure that E is always real. Substituting the stationary solution $\psi(\mathbf{x}, t) = \varphi(\mathbf{x})\exp(-i\omega_0 t)$ and the Galilei-transformed solution (3.3) into the last expression, we get correspondingly the energy of the stationary field

$$\mathrm{E}_0 = \omega_0\int\varphi^2\,d^3x + \int\left(G(\varphi^2) - \varphi^2G'(\varphi^2)\right)d^3x$$

and the energy of the "moving" field

$$\begin{aligned}\mathrm{E} &= \frac{i}{2}\int\left(\left(-i\omega\varphi'^2 - \varphi_k'\mathbf{v}\cdot\nabla\varphi_k'\right) - \left(i\omega\varphi'^2 - \varphi_k'\mathbf{v}\cdot\nabla\varphi_k'\right)\right)d^3x\\ &\quad + \int\left(G(\varphi'^2) - \varphi'^2G'(\varphi'^2)\right)d^3x\\ &= \omega\int\varphi^2\,d^3x + \int\left(G(\varphi^2) - \varphi^2G'(\varphi^2)\right)d^3x\end{aligned}$$

The integrals $\int\varphi^2\,d^3x$ and $\int(G(\varphi^2) - \varphi^2G'(\varphi^2))\,d^3x$ remain the same when φ is replaced with its space translate φ', that is, they do not depend on $\mathbf{v}$. Accordingly, one obtains that the energy change is proportional to the frequency change, and by expression (3.5), that it is proportional to the square of the velocity $\mathbf{v}$,

$$\mathrm{E} - \mathrm{E}_0 = (\omega - \omega_0)\int\varphi^2\,d^3x = (\omega - \omega_0)\langle\psi, \psi\rangle = -\frac{v^2}{4\mu}\langle\psi, \psi\rangle \tag{3.7}$$

Therefore, the energy difference $\mathrm{E} - \mathrm{E}_0$ *must be identified with the kinetic energy of the* ψ*-field, which is due to its translational motion, and the field must be endowed with a mass which is given by*

$$m = -\frac{1}{2\mu}\int\varphi^2\,d^3x = -\frac{1}{2\mu}\langle\psi, \psi\rangle, \qquad \text{with} \quad \mu < 0 \tag{3.8}$$

Since the kinetic energy cannot be negative, the constant μ must be chosen negative.

Expression (3.7) is a de Broglie-type relation in which the Hermitian norm $\langle\psi, \psi\rangle$ *plays the role of Planck's constant* $\hbar$ (both h and $\hbar$ will be called Planck's constant when no danger of ambiguity exists). We already saw that, since $\boldsymbol{\psi}$ is not normalized and the Hamiltonian functional is identified with the $\boldsymbol{\psi}$-field energy according to Proposition 1, the dimensionality of $\langle\boldsymbol{\psi}, \boldsymbol{\psi}\rangle$ for any CHE equation is *action*, which is the same as that of Planck's constant.

The relation between the kinetic energy and the frequency change can be derived without assuming a specific form for the CHE equation, like (3.1). It is sufficient to assume that the Lagrangian functional for the $\boldsymbol{\psi}$-field is Galilei-invariant. When the Hamiltonian density $\mathscr{H}[\boldsymbol{\psi}^*, \boldsymbol{\psi}]$ is known, the Lagrangian density $\mathscr{L}[\boldsymbol{\psi}^*, \boldsymbol{\psi}]$ for any CHE equations is given by expression (2.13). The Galilei transform of the stationary solution has the same form as (3.3), except that now the wavevector $\mathbf{k} = \mathbf{k}(\omega)$ is an unspecified function of ω. Taking this into account, the term with the time derivatives in $\mathscr{L}[\boldsymbol{\psi}^*, \boldsymbol{\psi}]$ gives

$$\frac{i}{2}\left(\boldsymbol{\psi}'^{\dagger}\frac{\partial\boldsymbol{\psi}'}{\partial t} - \frac{\partial\boldsymbol{\psi}'^{\dagger}}{\partial t}\boldsymbol{\psi}'\right) = \frac{i}{2}\left(\varphi_k'(\mathbf{v}\cdot\nabla\varphi_k') - (\mathbf{v}\cdot\nabla\varphi_k')\varphi_k'\right) + \omega\varphi'^2 = \omega\varphi'^2$$

Inserting the latter into (2.13), solving for $\mathscr{H}$, and integrating while noticing that $\boldsymbol{\varphi}'$ is a space translation of $\boldsymbol{\varphi}$, we find that the energy of the "moving" field is

$$\mathrm{E} = \int \mathscr{H}\, d^3x = \omega \int \varphi'^2\, d^3x - \int \mathscr{L}\, d^3x = \omega \int \varphi^2\, d^3x - \mathrm{L}$$

The energy of the stationary field, obviously, is

$$\mathrm{E}_0 = \omega_0 \int \varphi^2\, d^3x - \mathrm{L}_0$$

Here, L_0 and L are the values of the Lagrangian functional evaluated respectively with the stationary and with the Galilei-transformed solutions. By the Galilei invariance assumption, $\mathrm{L} = \mathrm{L}_0$; hence $\mathrm{E} - \mathrm{E}_0 = (\omega - \omega_0)\langle\boldsymbol{\psi}, \boldsymbol{\psi}\rangle$, which is the same de Broglie-type relation as in (3.7).

The de Broglie-type relation which connects the momentum $m\mathbf{v}$ of a nonrelativistic localized field with the wavevector $\mathbf{k}$ of its associated wave is obtained by expressing $\mu = -\langle\boldsymbol{\psi}, \boldsymbol{\psi}\rangle/2m$ from (3.8) and inserting it into the first of conditions (3.4),

$$\mathbf{v} = -2\mu\mathbf{k} = \frac{1}{m}\langle\boldsymbol{\psi}, \boldsymbol{\psi}\rangle\mathbf{k} \qquad \text{or} \qquad \mathbf{p} = m\mathbf{v} = \langle\boldsymbol{\psi}, \boldsymbol{\psi}\rangle\mathbf{k} \tag{3.9}$$

We see that the norm $\langle\boldsymbol{\psi}, \boldsymbol{\psi}\rangle$ appears again in place of Planck's constant. The following section provides independent evidence showing that the appearance of $\langle\boldsymbol{\psi}, \boldsymbol{\psi}\rangle$ in place of $\hbar$ is not coincidental.

4. THE QUANTUM COMMUTATOR

It is simple to show that even for certain nonlinear CHE equations the characteristics of the ψ-field as a whole, such as its position, velocity, and momentum, are given with real-valued bilinear functionals, and hence they are associated with Hermitian operators. For example, consider a localized solution ψ of equation (3.1). One can define the *position* $\mathbf{X} = (\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3)$ of the ψ-field, as a whole, with the functionals

$$\mathrm{X}_i = \frac{1}{\langle\psi, \psi\rangle}\int \psi^\dagger \psi x_i \, d^3x, \qquad \text{with} \quad i = 1, 2, 3 \tag{4.1}$$

This is the most natural way to specify the "center" of any distribution, regardless of the equations which describe it. Of course, (4.1) is identical with the expectation value of the *position operator* $\mathbf{x}$ in QM, except that the integral is divided by $\langle\psi, \psi\rangle$. However, no probabilistic interpretation will be attached to it, or to any other expectation values, since such would be entirely unnecessary.

To find the *velocity* of the ψ-field as a whole, we calculate the time derivative of X_i using the functional Poisson bracket (2.12),

$$\mathrm{V}_i = \frac{d\mathrm{X}_i}{dt} = \frac{1}{i\langle\psi, \psi\rangle}\int \left(\frac{\delta \mathrm{X}_i}{\delta\psi_k}\frac{\delta \mathrm{H}}{\delta\psi_k^*} - \frac{\delta \mathrm{H}}{\delta\psi_k}\frac{\delta \mathrm{X}_i}{\delta\psi_k^*}\right) d^3x = -\frac{2\mu}{i\langle\psi, \psi\rangle}\int \psi_k^* \frac{\partial\psi_k}{\partial x_i} \, d^3x$$

where H is given by (3.2). Hence, the velocity functional is associated with a Hermitian operator, whose form is identical with that of the *momentum* operator of quantum mechanics, even for some nonlinear evolution equations. This result can be generalized as follows. Consider two bilinear real-valued functionals

$$\mathrm{R} = \frac{1}{\langle\psi, \psi\rangle}\int \psi^\dagger \boldsymbol{R}\psi \, d^3x, \qquad \mathrm{S} = \frac{1}{\langle\psi, \psi\rangle}\int \psi^\dagger \boldsymbol{S}\psi \, d^3x \tag{4.2}$$

associated with the Hermitian operators $\boldsymbol{R}$ and $\boldsymbol{S}$. If they are to represent expectation values, their integrals must be divided by $\langle\psi, \psi\rangle$, since the ψ-field is a solution of a nonlinear equation, and hence it is not normalized. Then, it follows immediately from (2.10) that their infinite-dimensional Poisson bracket

$$\begin{aligned}\{\mathrm{R}, \mathrm{S}\} &= \frac{1}{i}\int \left(\frac{\delta \mathrm{R}}{\delta\psi_k}\frac{\delta \mathrm{S}}{\delta\psi_k^*} - \frac{\delta \mathrm{S}}{\delta\psi_k}\frac{\delta \mathrm{R}}{\delta\psi_k^*}\right) d^3x \\ &= \frac{1}{i\langle\psi, \psi\rangle^2}\int \left((\boldsymbol{R}\psi)^\dagger \boldsymbol{S}\psi - (\boldsymbol{S}\psi)^\dagger \boldsymbol{R}\psi\right) d^3x \\ &= \frac{1}{\langle\psi, \psi\rangle}\int \psi^\dagger \frac{\boldsymbol{RS} - \boldsymbol{SR}}{i\langle\psi, \psi\rangle}\psi \, d^3x\end{aligned}$$

is associated, precisely by the same rule (4.2), with the linear Hermitian operator

$$[R, S]_{\mathrm{QM}} = \frac{1}{i\langle\psi, \psi\rangle}(RS - SR) \tag{4.3}$$

An expression with the same form was obtained by Magri (1976), who followed a different path. However, the constant $\langle\psi, \psi\rangle$ does not enter in his result, because the functionals in his derivation are not normalized. Expression (4.3), of course, is the *quantum Poisson bracket* as Dirac (1958) called it, except that $\langle\psi, \psi\rangle$ appears in place of Planck's constant $\hbar$. Again!

By now, one recognizes that the seemingly trivial normalization procedure of quantum mechanics has a striking consequence when the field equations are nonlinear CHE and invariant under gauge type I transformations.

5. CONCLUSIONS

(a) It was shown that de Broglie-type relations hold for the space-localized singularity-free fields, whose form (3.3) as solutions of certain complex Hamiltonian evolution equations is

$$\psi(\mathbf{x}, t) = \varphi(\mathbf{x} - \mathbf{v}t)\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$$

where $|\varphi(\mathbf{x})| \to 0$ sufficiently fast when $|\mathbf{x}| \to \infty$. The complex wave $\exp i(\mathbf{k}\cdot\mathbf{x} - \omega t)$ satisfies all the requirements for a de Broglie wave [relations (3.7) and (3.9)]. Moreover, its group velocity $(\partial\omega/\partial k_1, \partial\omega/\partial k_2, \partial\omega/\partial k_3)$ obtained *directly* from the conditions (3.4)

$$\frac{\partial\omega}{\partial k_i} = -2\mu k_i = v_i, \qquad i = 1, 2, 3 \tag{5.1}$$

is equal to the region of localization velocity $\mathbf{v}$, which is the translational velocity of the *modulating* function $\varphi(\mathbf{x} - \mathbf{v}t)$.

There is a curious observation to be made here: The identity (5.1) holds regardless of how energy and mass are assigned to the ψ-field, and for any real value of the constant μ. Hence, it precedes the de Broglie-type relations, since the last are meaningless without these assignments. The same cannot be said for the corresponding identity in quantum mechanics, where it is derived from the de Broglie's relations.

(b) The de Broglie-type relation (3.7) applies, as already shown for the Galilei invariance case to the kinetic energy, not to the total energy. This is in a complete agreement with the corresponding de Broglie relation proper in nonrelativistic quantum mechanics. In general, the *total* energy of the ψ-field [given by (3.6)] is not proportional to the frequency ω, and its relation to ω is different for different nonlinearities, that is, for different functions $G(\psi^\dagger\psi)$.

(c) The uncertainty relation $\Delta x \Delta p \geq \hbar/2$ does not hold for the above solutions: While the "width" of the ψ-field region of localization Δx is finite, the "spread" in its momentum $\Delta p = \langle\psi, \psi\rangle\Delta k$ is zero, of course. In quantum mechanics, due to the linearity of its equations, free particles can be represented only by *wave packets*—obtained as Fourier integrals over $\mathbf{k}$. Hence, the "spread" in $\mathbf{k}$ (or in the momentum $\mathbf{p}$) is unavoidable. When the field equations are nonlinear, in place of the wave packets we have expressions like (3.3), with $\mathbf{k}$ and ω assuming well-defined single values. It is known that any wave packet dissipates itself with the passage of time, which is not really compatible with the concept of a particle. This is not necessarily the case with the solutions of nonlinear CHE equations.

Led by considerations entirely different from those above, Dirac concluded that the above relations, as understood now, are not fundamental: "I think one can make a safe guess that the uncertainty relations in their present form will not survive in the physics of the future" (Dirac, 1963, p. 49).

(d) The results of this paper are completely independent of the function $G(\psi^\dagger\psi)$, which determines the nonlinear term (provided that $\langle\psi, \psi\rangle$ is finite, of course). In view of the universality of the de Broglie relations, this is very satisfying. However, the freedom in choosing $G(\psi^\dagger\psi)$ does not mean that it can be set proportional to $\psi^\dagger\psi$, and thus allow the same arguments to be applied also to linear CHE equations. For, without the nonlinear term the value of $\langle\psi, \psi\rangle$ will be either undefined or divergent (for the singular solutions), which will destroy the validity of the above arguments.

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